

A parity map of framed chord diagrams

Denis Petrovich Ilyutko*, Vassily Olegovich Manturov†

Abstract

We consider framed chord diagrams, i.e. chord diagrams with chords of two types. It is well known that chord diagrams modulo 4T-relations admit Hopf algebra structure, where the multiplication is given by any connected sum with respect to the orientation. But in the case of framed chord diagrams a natural way to define a multiplication is not known yet. In the present paper, we first define a new module \mathcal{M}_2 which is generated by chord diagrams on two circles and factored by 4T-relations. Then we construct a “parity” map from the module of framed chord diagrams into \mathcal{M}_2 and a weight system on \mathcal{M}_2 . Using the map and weight system we show that a connected sum for framed chord diagrams is not a well-defined operation. In the end of the paper we touch linear diagrams, the circle replaced by a directed line.

1 Introduction

In topology and graph theory, many notions often have their “odd”, “non-orientable”, “framed” counterparts, see for example [6, 10, 11, 13, 17, 18, 19, 20, 21, 22, 23, 24]. Usually, even objects are better understood, however, in the odd case, it is much easier to catch the non-trivial information. The interrelation between the “even” and “odd” parts usually relies upon some functorial mappings, coverings etc., and allows one to understand better the real reason of various effects [21, 25, 26].

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It suffices to mention the three faces of the chord diagram theory: the Gauss diagram approach and the rotating circuit approach for four-valent graph [8, 9, 15, 16, 28], and J -invariants of plane curves [13]. Also framed chord diagrams are used for describing the combinatorics of a non-generic Legendrian knots in some 3-manifolds.

The most famous face of chord diagrams is the role they play in the chord diagram algebra [1, 2, 3, 4, 29]. The weight systems (i.e., linear functions on this algebra), due to Vassiliev–Kontsevich theorem, lead to Vassiliev invariants of knots, see [1, 3, 4]. Note that the multiplication in the chord diagram algebra is defined by taking a connected sum of two chord diagrams. This operation is well defined up to 4T-relation. In the case of framed chord diagrams one can define 4T-relations and consider a connected sum of two framed diagrams. An attempt to prove that this operation is well defined up to the 4T-relations fails.

The main goal of the present paper is to construct a “parity” map from the set of framed chord diagrams, factored by 4T-relations, to a set of objects where chords of only one type are used. Using this map and some invariant we demonstrate two examples of framed chord diagrams having different images under the map. Therefore, a connected sum is not a well-defined operation in the set of framed chord diagrams up to 4T-relations. Note that one “forgetful” map was defined in [12] for constructing framed weight systems.

The structure of the paper is as follows. In the next section we recall all necessary facts about framed chord diagrams. In Sec. 3 we introduce the notion of a double chord diagram, i.e. a chord diagram on two oriented circles, and define a weight system for the module generated by double chord diagram. Section 4 is devoted to a map from the framed chord diagrams module to the double chord diagrams module. Then, in Sec. 5 we apply this map and some weight system for proving that a connected sum of two chord diagrams is not a well-defined operation on the set of framed chord diagrams modulo 4T-relations. At the end of the paper we generalize all constructions for the case of linear diagrams, i.e. chord diagrams on a directed line instead of an oriented circle.

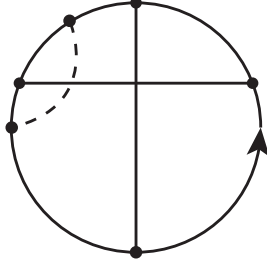


Figure 1: A framed chord diagram with two chords of framing 0 and a chord of framing 1

2 Framed chord diagrams

Throughout the paper, all graphs are finite. Let G be a graph with the set of vertices $V(G)$ and the set of edges $E(G)$. We say that a vertex $v \in V(G)$ has *degree* k if v is incident to k edges. A graph whose vertices have the same degree k is called *regular k -valent* or a *k -graph*. For any k , the free loop, i.e. the graph without vertices, is considered as a k -graph.

Definition 2.1. A *chord diagram* is a cubic graph consisting of a selected oriented Hamiltonian cycle (the *core circle*) and several non-oriented edges (*chords*) connecting points on the core circle in such a way that every point on the core circle is incident to at most one chord. A chord diagram is *framed* if a map (a *framing*) from the set of chords to $\mathbb{Z}/2\mathbb{Z}$ is given, i.e. every chord is endowed with 0 or 1.

Remark 2.1. We consider all framed chord diagrams up to orientation and framing preserving isomorphisms of graphs taking one core circle to the other one. In pictures the core circles of chord diagrams are oriented in counterclockwise manner. Chords having framing 0 are solid chords, and those having framing 1, are dashed ones, see Fig. 1.

Let M^f be the free \mathbb{Z} -module generated by all framed chord diagrams. Each element of M^f is a finite linear combination of framed chord diagrams with integer coefficients.

Definition 2.2. The module \mathcal{M}^f [13] of framed chord diagrams is the quotient module of M^f modulo the relations shown in Fig. 2. We refer to these relations as to *4T-relations*.

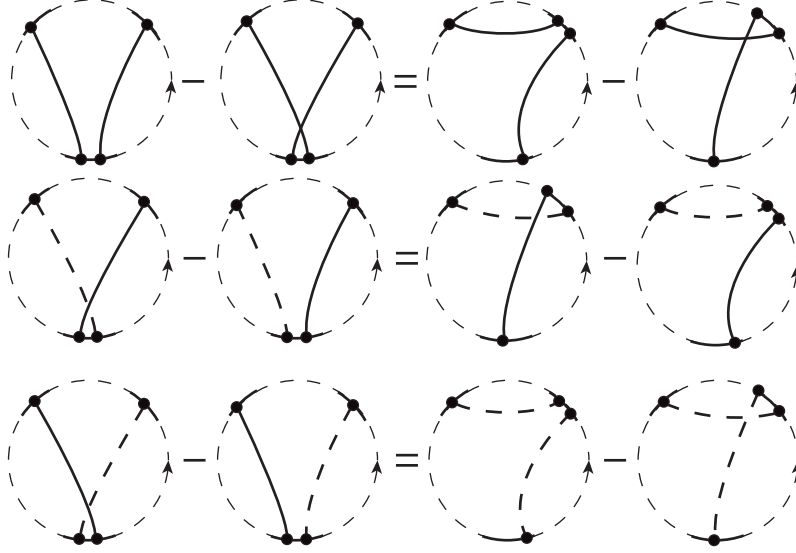


Figure 2: 4T-relations for framed chord diagrams

Remark 2.2. The pictures in Fig. 2 should be understood as follows. It is assumed that the endpoints of other chords can lie only in the dashed parts of the core circle and the combinatorial structure of chords not depicted in the pictures, is the same for all the four diagrams constituting the relation.

We can consider chord diagrams only with framings 0 and the corresponding 4T-relation. As a result we obtain the submodule \mathcal{M} of \mathcal{M}^f , see [12].

Considering the field \mathbb{R} instead of \mathbb{Z} we can obtain a commutative cocommutative Hopf algebra \mathcal{A} of chord diagrams [4]. The space \mathcal{A} is endowed with a natural product $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$, a natural coproduct $\Delta: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, the unit $e: \mathbb{R} \rightarrow \mathcal{A}$, the counit $\epsilon: \mathcal{A} \rightarrow \mathbb{R}$ and the antipode $\mathcal{S}: \mathcal{A} \rightarrow \mathcal{A}$. The multiplication is given by gluing two diagrams, i.e. a connected sum, and the coproduct is given by summing up the tensor products of pairs of chord diagrams formed by a decomposition of the set of chords into two complimentary subsets. Analogously, the space \mathcal{A}^f of framed chord diagrams is endowed with the comultiplication transforming it to the coassociative cocommutative coalgebra [12, 13].

3 Double chord diagrams

In this section we construct chord diagrams on two circles. Throughout this section all chords have the framing 1.

3.1 Basic definitions

Definition 3.1. A *double chord diagram* is a cubic graph consisting of two oriented disjoint circles (the *core circles*) and several non-oriented edges (*chords*) connecting points on the core circles in such a way that every point on a core circle is incident to at most one chord.

Remark 3.1. Double chord diagrams are considered up to isomorphisms of graphs preserving the orientations of two core circles.

Define the \mathbb{Z} -module M_2 as the set of finite \mathbb{Z} -linear combinations of double chord diagrams.

On the set of all double chord diagrams we can define relations analogous to the 4T-relation on chord diagrams without chords with framing 1. The difference in the definition of these moves is the following. In the case of double chord diagrams three pieces containing the endpoints of two singled chords may lie in the two core circles. We refer to these relations, see Fig. 3, as to *4T-relations*.

Definition 3.2. The module \mathcal{M}_2 of double chord diagrams is the quotient module of M_2 modulo the 4T-relations.

3.2 A weight system

We recall that a linear map from the space of framed chord diagrams satisfying 4T-relations is called a *weight system*.

Definition 3.3. We call any linear map from the space of double chord diagrams satisfying 4T-relations a *weight system* on the set of double chord diagrams.

Let us construct one example of a weight system on \mathcal{M}_2 .

Let us first define a *surgeries along the set of chords* of a double chord diagram, as it was done for chord diagrams and framed chord diagrams in see [2, 5, 8, 9, 13, 14, 27, 29, 30].

Let \mathcal{D} be a double chord diagram. For every chord belonging to one core circle we draw a parallel chord near it and remove the small

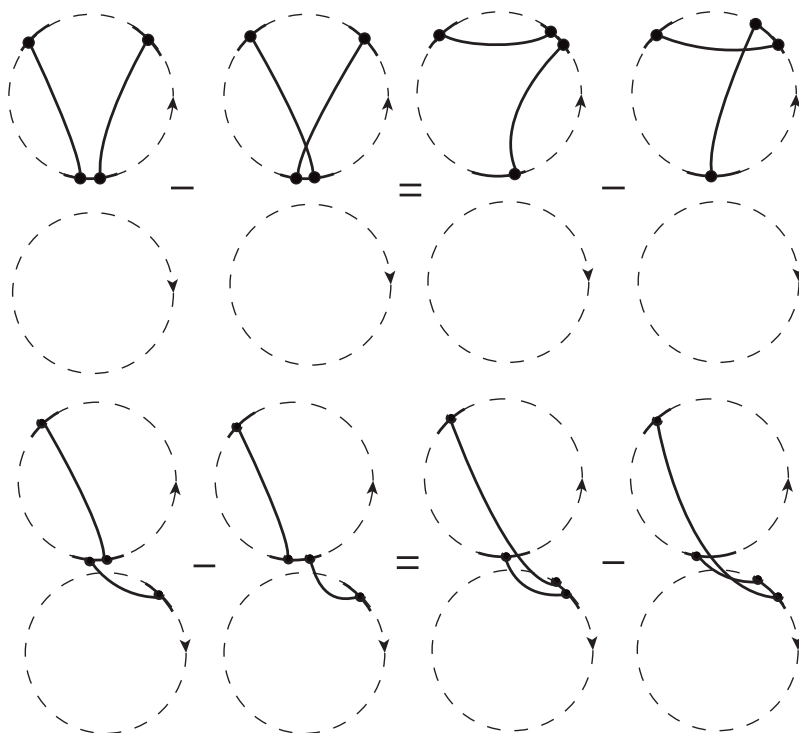


Figure 3: 4T-relations for double chord diagrams

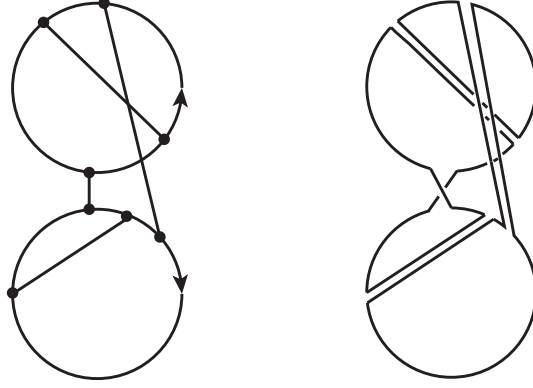


Figure 4: The surgery

arcs of the core circle between adjacent ends of the chords. For every chord with endpoints on two core circles we replace it with two chords in such a way that after removing the small arcs of the core circles between adjacent ends of the chords the orientations of the core circles are coherent, see Fig. 4.

By a small perturbation, the picture in \mathbb{R}^2 is transformed into a one-manifold $N(\mathcal{D})$ in \mathbb{R}^3 . Let $\beta_{\mathcal{D}}$ be the number of connected components of $N(\mathcal{D})$.

The proof of the following theorem is analogous to the proof of the corresponding theorem for framed chord diagrams, see [14].

Theorem 3.1. *Let \mathcal{D} be a double chord diagram. Then $\beta_{\mathcal{D}}$ is invariant under $2T$ -relations, see Fig. 5.*

Corollary 3.1. *The number $\beta_{\mathcal{D}}$ is invariant under $4T$ -relations.*

Define a map $w: \mathcal{M}_2 \rightarrow \mathbb{Z}$ by putting

$$w(\alpha_1 \mathcal{D}_1 + \dots + \alpha_k \mathcal{D}_k) = \alpha_1 \beta_{\mathcal{D}_1} + \dots + \alpha_k \beta_{\mathcal{D}_k},$$

where $\alpha_i \in \mathbb{Z}$. From Theorem 3.1 we see that w is a weight system on \mathcal{M}_2 .

4 A map from the module \mathcal{M}^f to the module \mathcal{M}_2

It turns out that there is a well-defined map ψ from the module \mathcal{M}^f to the module \mathcal{M}_2 .

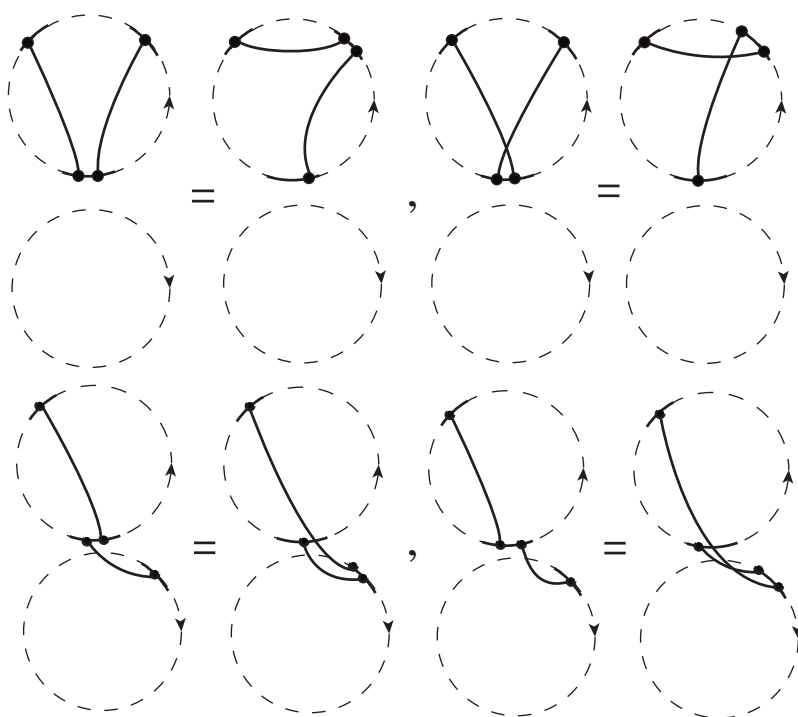


Figure 5: 2T-relations for double chord diagrams

Let us first define ψ on a framed chord diagram D with a core circle C and n chords. Construct double chord diagrams with the core circles C_1 and C_2 to be C as follows. For each chord d of D consider the two positions of it in C_1 and C_2 . Namely, if d has framing 0, then it can have its both endpoints either on C_1 or C_2 . If d has framing 1, then its endpoints lie on different core circles. This leads to 2^n possible choices. We define $\psi(D)$ to be the sum of these 2^n summands but the orientation of C_2 is reversed.

Then we extend the map ψ by linearity.

Example. Let D be a framed chord diagram depicted in Fig. 1. Then

$$\psi(D) = \psi \left(\begin{array}{c} \text{Diagram with two circles and two chords} \end{array} \right) = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \\ \text{Diagram 8} \end{array} + \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \\ \text{Diagram 11} \\ \text{Diagram 12} \end{array} + \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \\ \text{Diagram 15} \\ \text{Diagram 16} \end{array}.$$

Theorem 4.1. *The map $\psi: \mathcal{M}^f \rightarrow \mathcal{M}_2$ is well defined.*

Proof. We just have to check that ψ preserves the 4T-relations. We

consider only the third relation from Fig. 2. We have

$$\begin{aligned}
\psi \left(\text{Diagram 1} \right) &= \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5}, \\
\psi \left(\text{Diagram 6} \right) &= \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5}, \\
\psi \left(\text{Diagram 7} \right) &= \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5}, \\
\psi \left(\text{Diagram 8} \right) &= \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5}.
\end{aligned}$$

As a result we have

$$\begin{aligned}
\psi \left(\text{Diagram 1} - \text{Diagram 6} \right) &= \text{Diagram 2} - \text{Diagram 3} + \text{Diagram 4} - \text{Diagram 5} \\
&= \text{Diagram 2} - \text{Diagram 3} + \text{Diagram 4} - \text{Diagram 5} = \psi \left(\text{Diagram 7} - \text{Diagram 8} \right).
\end{aligned}$$

The other relations are checked analogously. \square

5 An application of ψ

It is well-known that the connection sum of chord diagrams is well defined [1]. In this section, by using ψ we show that a connected

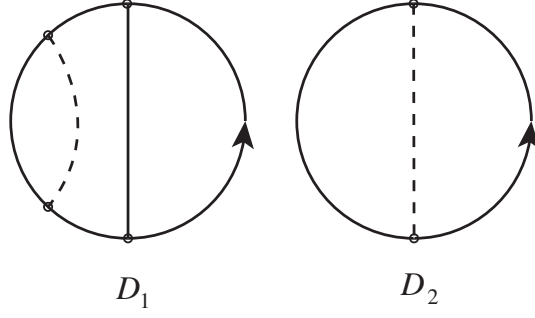


Figure 6: Two framed chord diagrams

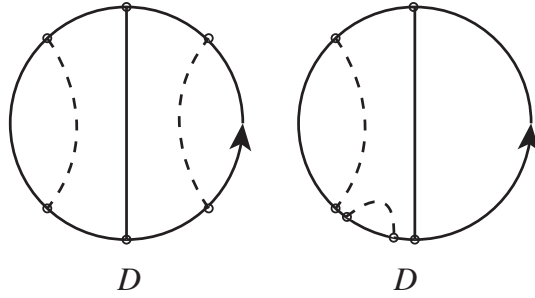


Figure 7: Two connected sums

sum of framed chord diagrams is not a well-defined operation in \mathcal{M}^f . Consider two framed chord diagrams D_1 and D_2 depicted in Fig. 6.

Choosing points on the chord diagrams in different ways we can obtain the following two connected sums D and D' , see Fig. 7.

We have

$$\begin{aligned}
 \psi(D) &= 4 \begin{array}{c} \text{circle with vertical solid chord} \\ \text{circle with vertical dashed chord} \end{array} + 4 \begin{array}{c} \text{circle with vertical dashed chord} \\ \text{circle with vertical solid chord} \end{array} = 8 \begin{array}{c} \text{circle with vertical solid chord} \\ \text{circle with vertical solid chord} \end{array}, \\
 \psi(D') &= 4 \begin{array}{c} \text{circle with horizontal solid chord} \\ \text{circle with horizontal dashed chord} \end{array} + 4 \begin{array}{c} \text{circle with horizontal dashed chord} \\ \text{circle with horizontal solid chord} \end{array} = 8 \begin{array}{c} \text{circle with horizontal solid chord} \\ \text{circle with horizontal solid chord} \end{array}.
 \end{aligned}$$

It is not difficult to show that the number of connected components of

$N \left(\begin{array}{c} \text{diagram 1} \\ \text{diagram 2} \end{array} \right)$ is equal to 1, but the number of connected components
of $N \left(\begin{array}{c} \text{diagram 3} \\ \text{diagram 4} \end{array} \right)$ equals 3. Therefore, $w(\psi(D)) = 8$ and $w(\psi(D')) = 24$
and the elements $8 \left(\begin{array}{c} \text{diagram 5} \\ \text{diagram 6} \end{array} \right)$ and $8 \left(\begin{array}{c} \text{diagram 7} \\ \text{diagram 8} \end{array} \right)$ do not coincide in \mathcal{M}_2 , so do
the elements D and D' in \mathcal{M}^f .

6 Linear diagrams

Besides framed chord diagrams and double chord diagrams, we can consider linear diagrams and double linear diagrams.

Definition 6.1. A *linear diagram* is an oriented line with a finite number of arcs having their endpoints on this line. A linear diagram is *framed* if a map (a *framing*) from the set of arcs to $\mathbb{Z}/2\mathbb{Z}$ is given, i.e. every arc is endowed with 0 or 1.

Remark 6.1. We consider all framed linear diagrams up to orientation and framing preserving isomorphisms of graphs taking one line to the other one. Arcs having framing 0 are solid arcs, and those having framing 1, are dashed ones, see Fig. 8.

Having a framed linear diagram G , we can construct the framed chord diagram, the closure $\text{Cl}(B)$ of G , by “closing” the line. It is not difficult to see that this operation (the map from the set of linear framed chord diagram to the set of framed chord diagram) is well defined.

Let L^f be the free \mathbb{Z} -module generated by all framed linear diagrams and \mathcal{L}^f be the quotient module of L^f modulo the relations shown in Fig. 3. We refer to these relations as to *linear 4T-relations*.

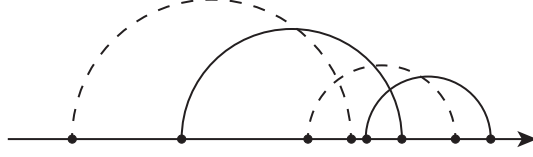


Figure 8: A framed linear diagram with two chords of framing 0 and two chords of framing 1

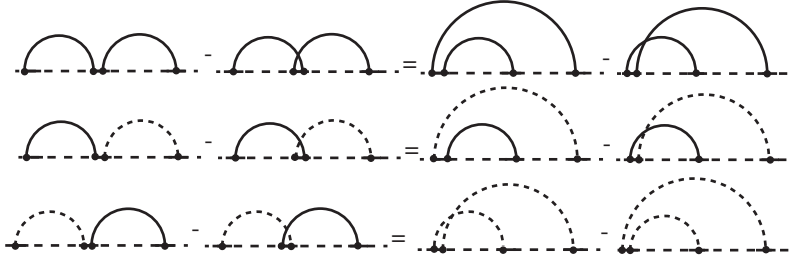


Figure 9: Linear 4T-relations for framed linear diagrams

Remark 6.2. In Fig. 9, the lines on the LHS and RHS of each equality are assumed to be oriented accordingly.

Analogously, we can consider chord diagrams only with framings 0 and the corresponding linear 4T-relations. As a result we obtain the submodule \mathcal{L} of \mathcal{L}^f . The module \mathcal{L} can be endowed with the structure of a commutative cocommutative Hopf algebra, where the a product $\mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{L}$ is given by gluing two lines according to the orientation.

Definition 6.2. A *double linear diagram* is a union of two disjoint framed linear diagrams and arcs with endpoints on both the oriented lines.

Remark 6.3. Double linear diagrams are considered up to isomorphisms preserving the orientations of lines.

Let L_2 be the free \mathbb{Z} -module of double linear diagrams and let \mathcal{L}_2 be the quotient module of L_2 modulo the relations (*4T-relations*) shown in Fig. 10.

Definition 6.3. Any linear map from the space of double linear diagrams satisfying linear 4T-relations is called a *weight system* on the set of double linear diagrams.

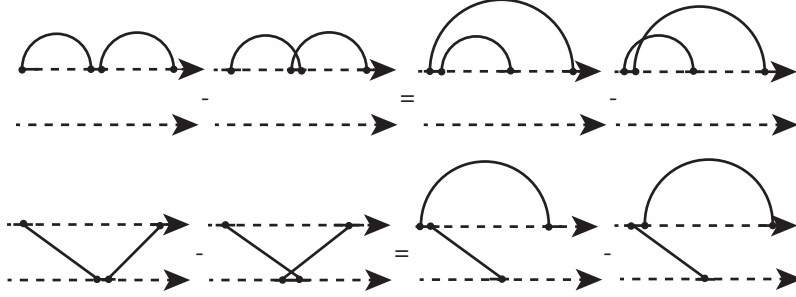


Figure 10: Linear 4T-relations for double linear diagrams

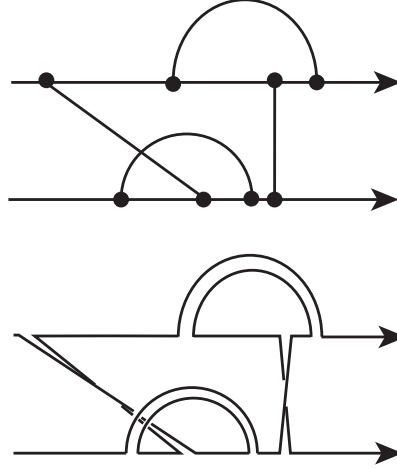


Figure 11: The surgery

Let \mathcal{G} be a double chord diagram. Analogously to Sec. 3.2 we can define the *surgery* for \mathcal{G} and obtain a one-manifold $N(\mathcal{G})$, see Fig. 11. $N(\mathcal{G})$ consists of two lines and a collection of circles.

Theorem 6.1. *The number $\beta_{\mathcal{G}}$ of connected components of $N(\mathcal{G})$ is invariant under linear 2T-relations, see Fig. 12.*

Corollary 6.1. *The number $\beta_{\mathcal{G}}$ is invariant under linear 4T-relations.*

Define a map $w_l: \mathcal{L}_2 \rightarrow \mathbb{Z}$ by putting

$$w_l(\alpha_1 \mathcal{G}_1 + \dots + \alpha_k \mathcal{G}_k) = \alpha_1 \beta_{\mathcal{G}_1} + \dots + \alpha_k \beta_{\mathcal{G}_k},$$

where $\alpha_i \in \mathbb{Z}$. Using Theorem 6.1 we get that the map w_l is a weight system on \mathcal{L}_2 .

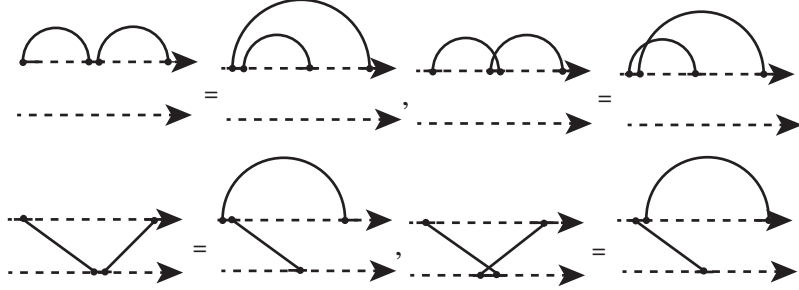


Figure 12: Linear 2T-relations for double linear diagrams

Let us construct a map $\psi_l: \mathcal{L}^f \rightarrow \mathcal{L}_2$ in the same way as it was done in Sec. 4. For example,

$$\psi_l \left(\text{diagram with two arcs} \right) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} + \text{diagram 5} + \text{diagram 6} + \text{diagram 7} + \text{diagram 8} + \text{diagram 9} + \text{diagram 10}.$$

Theorem 6.2. *The map $\psi_l: \mathcal{L}^f \rightarrow \mathcal{L}_2$ is well defined.*

Having two linear framed diagrams G_1 and G_2 we can define their connected sum by gluing one line to the other one. For example,

$$\text{if } G_1 = \text{diagram 1} \text{ and } G_2 = \text{diagram 2}, \text{ then } G_1 \# G_2 = \text{diagram 3} \text{ and } G_2 \# G_1 = \text{diagram 4}.$$

Opposite to the case of framed chord diagrams, we have the following theorem.

Theorem 6.3. *For any two linear framed diagrams we have*

$$w_l(\psi_l(G_1 \# G_2)) = w_l(\psi_l(G_2 \# G_1)).$$

Proof. Let us mark out the lines in each summand of $\psi_l(G_1)$ and $\psi_l(G_2)$. Then each summand of $\psi_l(G_1 \# G_2)$ and $\psi_l(G_2 \# G_1)$ is the connected sum of a summand from $\psi_l(G_1)$ and a summand from $\psi_l(G_2)$, where the first (second) line is connected with the first (second) one. The validity of the theorem follows from the following fact.

Let \mathcal{G}_1 and \mathcal{G}_2 be two double linear diagrams with marked lines. Then $\beta_{\mathcal{G}_1 \# \mathcal{G}_2}$ is equal to $\beta_{\mathcal{G}_1} + \beta_{\mathcal{G}_2} - 1$ or $\beta_{\mathcal{G}_1} + \beta_{\mathcal{G}_2} - 2$, where 1 and 2 depend on the initial diagram \mathcal{G}_1 and \mathcal{G}_2 , but do not depend on the connected sums. This statement can be easily proved by analyzing possible connections of non-compact components of $N(\mathcal{G}_1)$ and $N(\mathcal{G}_2)$. \square

Acknowledgments

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